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# Effect of gravity waves on turbulence decay in stratified fluids

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A theoretical calculation is made of the decay of turbulence energy in the presence of coherent internal gravity waves of various intensities. The wave–turbulence interaction considered is energy production by wave shear. The production term (stress) is calculated by a second-order closure, with temperature fluctuations accounted for by buoyancy subrange theory. This formalism applies to large or small turbulence Froude number, both extremes of which are often encountered in experimental turbulence decay. The theoretical turbulence decay is shown to be a universal function of the wave shear (strain rate) and wave frequency provided that the energy is expressed in terms of the buoyancy wavenumber $k_B$, and time is expressed in terms of $N$, the Brunt–Väisälä frequency. With the amplitude of wave shear characterized by a gradient Richardson number $Ri_0$, the turbulence decay is found to undergo a sudden transition from rapid decay to a much slower oscillating decay when $Ri_0$ is less than about 0.4. The transition time occurs at about $t \approx 2\pi N^{-1}$. If $Ri_0$ exceeds 0.8 the rate of decay exceeds that of a neutral fluid. A transition in turbulence decay was observed in experiments by Dickey & Mellor (1980). It might explain the continued presence of turbulence in dynamically stable regions of oceans or atmosphere. The theory is compared in much detail with the Dickey & Mellor experiment. A briefer comparison is also made with other experiments, and with previous calculations of turbulence maintenance by steady mean shear. A simple explanation is proposed of why a transition is observed in a vertical grid experiment but not in horizontal grid experiments.

## 1. Introduction

Recently, Dickey & Mellor (1980, hereinafter referred to as DM) conducted experiments of decaying turbulence in neutral and stratified fluids. As the turbulence energy was allowed to decay in a stratified fluid, they found a transition in the energy decay rate in which the decay was slowed down and oscillated with time. Related turbulence transitions have been observed previously (e.g. Pao 1973; Lin & Pao 1979; Lin & Veenhuizen 1974). The new aspect of the DM experiments is that a sharp transition was found in the energy decay law. (Previously a transition was observed visually and in shadowgraphs but not in the measured energy decay.)

Similarly appearing sinusoidal oscillations of turbulence intensity have also been observed by radar observations of the atmosphere (Van Zandt, Green & Clark 1979; Rüster, Rottger & Woodman 1978). DM characterized the observed turbulence decay as a transition from a turbulence regime to a coherent gravity-wave regime. To calculate the transition time, they hypothesized a decreased dissipation rate, which implies an interruption of the cascade process by the gravity waves. An alternative mechanism is simply that turbulence and gravity waves coexist, with the waves 'feeding' energy into the turbulence; i.e. turbulence production due to wave shear. Wave-shear–turbulence interactions are known to occur in the atmosphere, and one wonders to what extent such an interaction can quantitatively explain the turbulence decay observed by DM, or in other turbulence experiments for that matter.

The purpose of this paper, then, is to calculate the production of turbulence by gravity-wave shear, and the influence of this production on turbulence decay. In this calculation, the stress (momentum flux) which occurs in the production term must be formulated with sufficient care to account for both large and small Froude number, the ratio of turbulence kinetic energy to potential energy. Both extremes are generally encountered by decaying turbulence. An eddy-diffusion coefficient of the typical form $\alpha v_0 k_0^{-1}$ (where $\alpha$ is an empirical constant, $v_0$ is the variance of turbulence velocity and $k_0^{-1}$ is a scale length of the turbulence) is not appropriate to these extremes because the scale length $k_0^{-1}$ varies greatly with Froude number and, if uncorrected for, causes an overestimate of the rate of wave shear production. Instead, a second-order closure is used for the stress with temperature fluctuations accounted for by buoyancy subrange theory. This theory allows us to dispense with the temperature equation (thermodynamic equation). The formulation is equivalent to standard second-order closures, and its relative simplicity permits an analytic expression for the stress to be derived. The derived stress is expected to have the same form as an eddy-viscosity model, but with a viscosity coefficient that applies to arbitrary Froude number.

With this formulation, the turbulence decay is calculated for various wave frequencies and amplitudes. The theoretical decay is afterwards compared with that observed by DM, and more briefly with other experiments. A comparison is also made with previous considerations of turbulence maintenance by steady mean shear (Monin & Yaglom 1971) – the limit of zero wave frequency. Attention will also be given to the puzzling question raised by DM of why a sudden transition in turbulence decay was not observed with a horizontally towed grid experiment (e.g. Lin & Veenhuizen 1974) but was observed with their vertically towed grid.

Although our wave turbulence calculation was motivated by the DM experiment, we hope it will be of geophysical interest as well. Gravity waves and turbulence are known to coexist in oceans and atmospheres, where their interactions are of fundamental significance (e.g. Gargett *et al.* 1981; Caldwell *et al.* 1980; Dutton & Panofsky 1970; Boucher 1974; Einaudi, Lalas & Perona 1978/79; Van Zandt *et al.* 1979 – see their figure 1; Rüster *et al.* 1978). However, the details of these interactions are difficult to measure in a geophysical environment. For this reason laboratory investigations can often be of especial importance.

Very recently, Fua *et al.* (1982) considered a gravity-wave–turbulence interaction for the interesting case where the background gradient Richardson number is slightly larger than $\frac{1}{4}$ and the wave amplitude small. Consideration was given to the more elaborate problem of turbulence 'triggering' by a small-amplitude wave, rather than to turbulence decay in the presence of large-amplitude waves. In their formalism, Fua *et al.* considered a 'feedback' of turbulence on wave, which we do not, and described wave–shear interaction by the eddy-viscosity coefficient $\alpha v_0 k_0^{-1}$.

The organization of the paper is as follows. A theoretical formulation of the wave shear production is given in §2.1. The principal result is a dimensionless equation that determines the temporal evolution of turbulence kinetic energy density. A brief comparison is made with laboratory and atmospheric observations in §2.2, and a detailed comparison is made with the DM experiment in §3. Section 4 contains a brief discussion of turbulence maintenance, and §5 is a summary.

## 2. Turbulence decay in the presence of gravity waves

### 2.1. *Theory*

Our aim is to solve the energy-balance equation for turbulence decay in the presence of wave-shear production. The energy-balance equation (Reynolds-stress equation) is approximately given by

$$\left(\frac{\partial}{\partial t}+\boldsymbol{U}_0\cdot\nabla\right)\left(\frac{q^2}{2}\right)=-\langle\boldsymbol{u}'\cdot(\boldsymbol{u}'\nabla\boldsymbol{U}_0)\rangle+\frac{\langle\rho'\boldsymbol{u}'\rangle\cdot\nabla P_0^0}{\rho_0^2}-\epsilon_\nu, \tag{1}$$

where $\mathbf{U}_0$ is the mean velocity of the coherent gravity waves, $\boldsymbol{u}'$ is the random fluctuation velocity (turbulence velocity), $\frac{1}{2}q^2\equiv\frac{1}{2}\langle\boldsymbol{u}'\cdot\boldsymbol{u}'\rangle$ is the turbulence kinetic energy, $\rho_0$ is the average particle density, $\rho'$ is the fluctuation density, $P_0$ is the average pressure, $\epsilon_\nu$ is the energy dissipation rate due to molecular viscosity, and the angular brackets denote an ensemble average. Correlations of triple-velocity and pressure–velocity fluctuations (turbulent diffusion) are neglected under the assumption that the fluctuations are approximately homogeneous. That assumption could be relaxed.

The first term on the right-hand side of (1) is the production of turbulence by wave shear $\nabla\boldsymbol{U}_0$, the second term is the loss of turbulence by buoyancy, and the last term is the loss of turbulence by molecular dissipation.

For simplicity, we consider cylindrical symmetry (with vertical coordinate $z$ and horizontal coordinate $H$), and ignore the azimuthal components of $\boldsymbol{U}_0$, so that the problem is effectively reduced to two coordinates ($z$ and $H$). Consideration is given to only the mean shears $\partial W/\partial H$ and $\partial U_H/\partial z$, where $W$ and $U_H$ denote respectively the vertical and horizontal velocities of the coherent gravity waves. Equation (1) then reduces to

$$\frac{\partial}{\partial t}\frac{q^2}{2}=-\langle u_z'u_H'\rangle\left(\frac{\partial W}{\partial H}+\frac{\partial U_H}{\partial z}\right)-\frac{\langle\rho'\boldsymbol{u}'\rangle\cdot\nabla P_0^0}{\rho_0^2}-\epsilon\nu, \tag{2}$$

where $u_z'$ and $u_H'$ are respectively the vertical and horizontal components of $\boldsymbol{u}'$.

The main task is to express the terms on the right-hand side of (2) as functions of $q^2$ and $\nabla\boldsymbol{U}_0$, so as to obtain a closed equation for $q^2$. The key term to be derived is $\langle u_z'u_H'\rangle$.

This term is derived in the Appendix by means of a standard second-order closure combined with buoyancy subrange theory. The resulting expression for $\langle u_z'u_H'\rangle$ agrees in form with an eddy-viscosity model and is given by

$$\langle u_z'u_H'\rangle=-\frac{\tau_L q^2}{3}\left(\frac{\partial W}{\partial H}+\frac{\partial U_H}{\partial z}\right), \tag{3}$$

$$\tau_L=\frac{0.35}{3^{\frac{1}{2}}}\frac{k_0q}{\frac{1}{3}k_0^2q^2+0.8N^2}, \tag{4}$$

where $k_0$ is a characteristic wavenumber of the energy-containing region of the spectrum defined by (A 13), $N\equiv(g\rho_0^{-1}\partial\rho_0/\partial z)^{\frac{1}{2}}$ is the Brunt–Väisälä frequency, and $\tau_L$ is the Lagrangian timescale of a stratified fluid. This timescale applies for all $k_0q/N$ (turbulent Froude number). Note that $\tau_L$ properly reduces to the correct neutral limit as $N \to 0$ (e.g. Tennekes & Lumley 1972). It also reduces to the correct strong-stratification limit as $N \to \infty$, in agreement with Lilly, Waco & Adlefang (1974), Caldwell *et al.* (1980) and Weinstock (1978*b*).

The difference between (3) and an eddy-viscosity formulation is that, instead of $\frac{1}{3}\tau_L q^2$, the latter has an eddy-viscosity coefficient given by $\alpha k_0 q$, where $\alpha$ is an empirical constant. Such an eddy coefficient does not fully account for stable stratification and overestimates the rate of mean-flow–turbulence energy transfer.

The second term on the right-hand-side of (2), the buoyancy-flux term, is given in the Appendix by (A 12) as

$$\langle \rho' \boldsymbol{u}' \rangle \cdot \boldsymbol{\nabla} P_0 / \rho_0^2 = -\tfrac{1}{3}\tau_L q^2 N^2, \tag{5}$$

and $\epsilon_\nu$ is given by (A 14) in terms of $q$ and $k_0$. On substituting (3) and (5) into (2), the energy-balance equation becomes

$$\frac{\partial}{\partial t}\frac{q^2}{2} = \tau_L \frac{q^2}{3}\left[\left(\frac{\partial W}{\partial H} + \frac{\partial U_H}{\partial z}\right)^2 - N^2\right] - \epsilon_\nu, \tag{6}$$

where $\tau_L$ is given by (4) and $\epsilon_\nu$ by (A 14). This equation differs from previous ones only in that $\tau_L$ is given by (4), which is appropriate for stratified fluids. This equation could also have been obtained by a standard second-order turbulence model, with perhaps different values of the numerical constants in $\tau_L$. The use of buoyancy subrange theory merely allowed us to dispense with the temperature equation. The fairly simple-looking appearance of (6) is due to our simplifying conditions of quasi-homogeneity and small anisotropy. The anisotropy is maintained small by the coherent gravity waves which, we will see, retard the energy decay (small anisotropy of kinetic energy is observed by DM). By means of these simplifications we are able to treat stratification in a more rigorous fashion and isolate its influence on turbulence decay.

We next show that (6) can be non-dimensionalized to obtain a universal equation for wave-shear–turbulence interactions. To do so, it is convenient to first express the mean strain rate in terms of gravity-wave parameters as follows:

$$\frac{\partial W}{\partial H} + \frac{\partial U_H}{\partial z} = S_0(\boldsymbol{r}) \cos(\sigma t + \phi), \tag{7}$$

where $S_0(\boldsymbol{r})$ denotes the amplitude of the mean strain rate at position $\boldsymbol{r}$, $\sigma$ is the frequency of the wave, and $\phi$ its phase. In general, $S_0(\boldsymbol{r})$ is the sum of many Fourier components $S_0(\boldsymbol{k}) \equiv \int \mathrm{d}\boldsymbol{r}\, S_0(\boldsymbol{r}) \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})$, and may be spatially periodic. It is also useful to express the mean strain rate in terms of the gradient Richardson number $Ri$ by

$$\left.\begin{aligned} Ri &= N^2\left(\frac{\partial W}{\partial H} + \frac{\partial U_H}{\partial z}\right)^{-2} \equiv Ri_0 \cos^{-2}(\sigma t + \phi), \\ Ri_0 &\equiv N^2 S_0^{-2}, \end{aligned}\right\} \tag{8}$$

and of course $Ri$ oscillates with time because $\partial W/\partial H + \partial U_H/\partial z$ does. Here, $Ri_0$ is the amplitude of the oscillation. We treat the amplitude magnitude $Ri_0$ as a variable parameter to see how it influences the theoretical decay of $q^2$. The energy decay is now obtained in terms of gravity-wave parameters, and $k_0$ by substitution of (4), (7) and (A 14) into (6):

$$\frac{\partial}{\partial t}\frac{q^2}{2} = \frac{0.067 k_0 q^2 N^2}{\frac{1}{3}k_0^2 q^2 + 0.8 N^2}\left[Ri_0^{-1}\cos^2(\sigma t + \phi) - 1\right] - 0.083 k_0 q^3. \tag{9}$$

Finally, we reduce (9) to a dimensionless equation in terms of a dimensionless length and time. We are encouraged to do so because there is reason to believe that stably stratified turbulence has a universal behaviour when the turbulence-energy length $k_0^{-1}$ is expressed in terms of the buoyancy wavenumber $k_B \equiv Nq^{-1}3^{\frac{1}{2}}$ and time is expressed in terms of the Brunt–Väisälä frequency (e.g. Gargett *et al.* 1981; Weinstock 1978*a*; Lumley 1964). The dimensionless length is $3^{\frac{1}{2}}Nq^{-1}k_0^{-1} \equiv k_B/k_0$, which can also be viewed as a dimensionless r.m.s. velocity since it depends on $q$. We use the simplifying notation $Z \equiv 3^{\frac{1}{2}}Nq^{-1}k_0^{-1}$. The dimensionless time is $Nt$. To express (9) in terms of these dimensionless scales, we divide both sides by $k_0 q^3$ and replace $t$ by $Nt/N$ to obtain

$$-(k_0^{-1}q)\frac{\partial}{\partial(Nt)}\frac{Nq^{-2}}{2} = \frac{0.067}{Z^{-2}+0.8}\left[Ri_0^{-1}\cos^2\left(\frac{\sigma}{N}Nt+\phi\right)-1\right]-0.083. \tag{10}$$

The non-dimensionalization of (10) would be complete if $k_0/q$ were constant, or slowly varying compared with $q^2$, since then we would have

$$k_0^{-1}q\frac{\partial}{\partial(Nt)}(Nq^{-2}2^{-1}) = \frac{\partial}{\partial(Nt)}(Nq^{-1}k_0^{-1}2^{-1}) \equiv 3^{-\frac{1}{2}}2^{-1}\frac{\partial}{\partial(Nt)}Z.$$

The near constancy of $k_0/q$ has been well documented for the initial stages of turbulence decay in neutral fluids (e.g. Batchelor & Townsend 1948; DM), where it is found that $q^2 \propto t^{-1}$, and $k_0^2 \propto t^{-1}$, so that

$$k_0 = Aq, \quad A \approx \text{constant}.$$

For our case of stratified fluids in the presence of coherent gravity waves DM find that $k_0/q$ is nearly constant throughout the relatively long duration of their experiment of several Brunt–Väisälä periods. We will assume in (10) that $k_0/q$ is relatively slowly varying for turbulence decay over the first several Brunt–Väisälä periods. In that case, (10) reduces to the desired dimensionless energy-decay equation

$$-\frac{\partial Z}{\partial(Nt)} = \frac{0.23}{Z^{-2}+0.8}\left[Ri_0^{-1}\cos^2\left(\frac{\sigma}{N}Nt+\phi\right)-1\right]-0.29, \tag{11}$$

$$Z \equiv 3^{\frac{1}{2}}Nq^{-1}k_0^{-1} = 3^{\frac{1}{2}}A^{-1}Nq^{-2}.$$

This surprisingly simple equation shows that turbulence decay in a stratified fluid in the presence of coherent gravity waves is a universal function of the wave Richardson number $Ri_0$ and the dimensionless wave frequency $\sigma/N$ – provided that the r.m.s. turbulence energy is expressed in terms of $Nk_0^{-1}$ and the time expressed in terms of $N$.

The solution of (11) for $Z$ is easily obtained by computer and for various values of $Ri_0$ and $\sigma/N$. The solutions for $\sigma/N = 0.65$ with $\phi = 0$ are given in figure 1, $\sigma/N = 0.325$ with $\phi = -\frac{1}{2}\pi$ in figure 2, and $\sigma = 0$ with $\phi = 0$ in figure 3. These figures show that the behaviour of the turbulence decay depends strongly on the value of $Ri_0$. The most distinct or unusual behaviour occurs for the range $Ri_0 < 0.4$. In this range, the initially rapid decay suddenly decreases at $t/2\pi \sim 0.7N^{-1}$, and thereafter oscillates with period $2\sigma$. A very similar behaviour was observed by DM (see their figure 6). We will discuss this particularly interesting case in §3 and examine whether or not it can explain the DM experiment – the experiment that motivates this communication. First let us discuss general features of the theoretical decay, and its relation to other experiments.

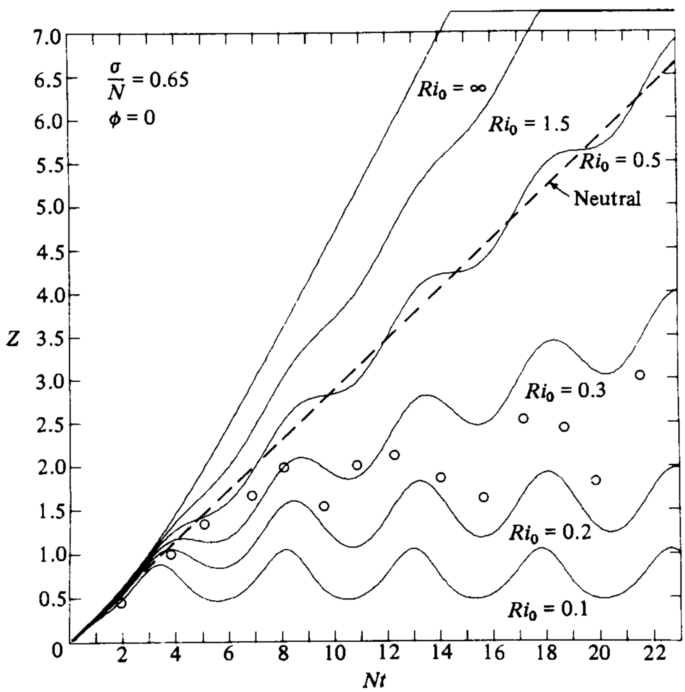


FIGURE 1. Theoretical kinetic-energy decay in a stratified fluid showing $Z$ versus $t$ for various values of $Ri_0$ with $\sigma/N = 0.65$ and $\phi = 0$. The dashed line is for a neutral fluid. The points are experimental data taken from Dickey & Mellor (1980).

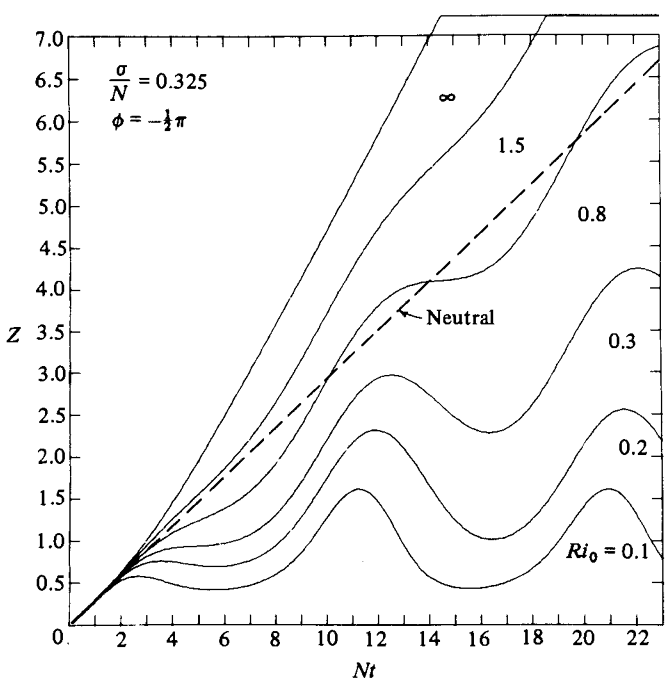


FIGURE 2. Same as figure 1 with $\sigma/N = 0.325$ and $\phi = \frac{1}{2}\pi$.

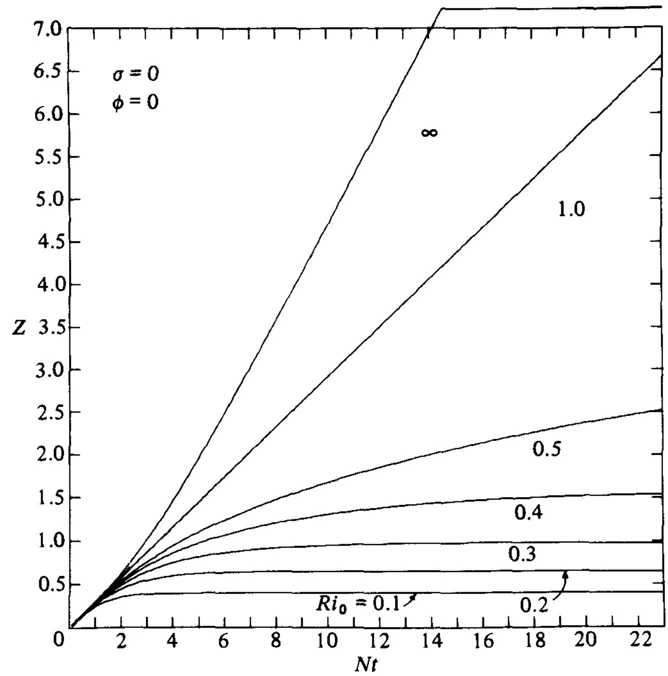


FIGURE 3. Same as figure 1 with $\sigma/N = 0$ and $\phi = 0$.

### 2.2. *Relation to experiment*

The behaviour of turbulence decay shown in figure 1 can be divided into three different classes as follows. For $Ri_0 < 0.4$ the turbulence decay undergoes a sudden transition near $(2\pi)^{-1}t \sim N$ from rapid monotonic decay to slow oscillating decay; for $0.4 < Ri_0 < 0.8$ the decay is close to neutral decay throughout the entire time interval, and resembles the decay observed by Britter *et al.* (1983); and for $Ri > 0.8$ the decay is more rapid than in neutral fluids, and resembles the stratified decay usually found in horizontal-grid experiments (e.g. Lin & Pao 1979; Lin & Veenhuizen 1974). It is evident that the existence of a transition depends critically on the value of $Ri_0$. If $Ri_0$ exceeds 0.8 the decay is hardly influenced by coherent gravity waves. Such may be the case of previous experiments that used a horizontally towed grid.

Hence, from a theoretical point of view, the different kinds of experimental decay could be accounted for by differences in the amplitudes $Ri_0$ of coherent gravity waves. In particular, this (wave-amplitude) criterion provides a straightforward explanation of the differences observed between vertically and horizontally towed grid turbulence, and is given special attention at the end of §5.

With regard to the experiment of Lin & Pao, it is seen in their figure 2 that $q^2$ decays faster than $t^{-1}$ during the initial interval. A similar behaviour is seen in our theoretical figures for $Ri_0 \gtrsim 1.5$. An explanation is that the faster-than-$t^{-1}$ decay is due to the decay of $\langle w^2\rangle$, the vertical component of $q^2$, which decays faster than the horizontal components $\langle u^2\rangle$ and $\langle v^2\rangle$. Hence $q^2 = \langle u^2\rangle + \langle v^2\rangle + \langle w^2\rangle$ decays faster than $t^{-1}$ during the initial period where the magnitude of $\langle w^2\rangle$ is still comparable to $\langle u^2\rangle$. After the initial period, however, $\langle w^2\rangle$ becomes much smaller than $\langle u^2\rangle$, and no longer contributes much to the decay. The decay rate then slows down to that of $\langle u^2\rangle + \langle v^2\rangle$. This appears so in experiment and theory. However, our calculated decay must be viewed with caution, because the assumption of small anisotropy in Appendix A breaks down when $Ri_0 \gtrsim 0.8$ and $Nt \gtrsim 2$. This assumption is only justifiable when the gravity waves are sufficiently intense to retard the decay of $\langle w^2 \rangle$.

With regard to radar measurements of the atmosphere by Van Zandt *et al.* (1979 – see their figure 1) and Rüster *et al.* (1978), their observations of sinusoidal modulations of turbulence intensity strongly resemble figure 1 with $Ri_0 \approx 0.1$. This small $Ri_0$ value corresponds to a large wave shear and implies that a strong instability must have occurred.

## 3. Comparison with the Dickey–Mellor experiment

It is very tempting to compare our theory with the experiments of DM because the theoretical curves very closely resemble their observations (their figures 4–6). However, this comparison is complicated by the fact that, partly, the measured fluctuations may be coherent gravity waves of small wavelength and, partly, random fluctuations (turbulence). The observations cannot distinguish between the two because the measurements are averaged over a volume whose dimensions are about 12.7 $cm^3$. (This spatial averaging will remove gravity wavelengths as large as 12.7 cm from the mean motion.) Let us then compare theory with experiment under the assumption that only a small part of the measured fluctuations are short-scale coherent gravity waves, and, afterwards, discuss the implications of that assumption.

Upon making this comparison, it can be seen that the theoretical decay for $Ri_0 \approx 0.25$ in figure 1 agrees with the experimental measurements (figure 6 of DM) in much detail. The value of $\sigma/N$ in these figures was chosen equal to the experimental value 0.65. The main features of the agreement are: (i) a sudden decrease (transition of decay rate; (ii) the approximate time $Nt$ at which the transition occurs; (iii) the mean value of the slope (decay rate) in the internal-wave region (large-$Nt$ region); and (iv) the amplitude of the $q^2$ oscillation. (For the quantitative comparison in items (iii) and (iv) we used the experimental value $A \equiv k_0/q = 0.605$ s $cm^{-2}$. This value is determined by noting that DM use the expression $\epsilon_\nu = q^3/A$, and $A$ is a scale that they experimentally relate to the integral scale $L$ by $A = 4.6$. Comparison of their $\epsilon_\nu$ expression with (A 13) gives $k_0 = 2.6L^{-1}$. They also give a relation between $L$ and $q$ as $qL = J = 4.3$ $cm^2/s$. Hence we have $k_0/q = 2.6/4.3$ s $cm^{-2} = 0.605$ s $cm^{-2}$. We also use the experimental value $N = 0.378$ $s^{-1}$. The relation between $Z$ and $(\frac{1}{3}q^2)^{-1}$ for that experiment is $Z = 0.361(\frac{1}{3}q^2)^{-1}$, with $q$ in cm $s^{-1}$.) We hasten to add that the value $Ri_0$ here has nothing to do with the linear stability condition $Ri = \frac{1}{4}$. The present value arises from nonlinear considerations, and the fact that the $Ri$-values are the same is a coincidence. There is a wide range of $Ri_0$ for which the transition occurs, and the experimental value could have had any of these.

The preceding value $Ri_0 = 0.25$ was obtained by matching the mean slope of the theoretical $Z$-curves ($q^{-2}$ curves) with the mean slope of the observed $q^{-2}$ (figure 6 of DM) in the large-$t$ region. An independent estimate of $Ri_0$ can be determined by comparing the oscillation amplitude seen in figure 6 of DM with the theoretical amplitude. An analytical expression for the latter amplitude is readily derived from (11) at large $Nt$ because then $Z^{-2}$ can be neglected in the denominator on the right-hand side.

Straightforward integration of (11) then yields

$$-Z(t) = 0.29\left(\frac{1}{2Ri_0}-2\right)Nt+\frac{0.29}{4Ri_0(\sigma/N)}\sin\left(2\frac{\sigma}{N}Nt+\phi\right), \tag{12}$$

and consequently the oscillation amplitude is $0.29(4Ri_0\sigma N^{-1})^{-1}$. Note that (12) provides two independent ways to determine $Ri_0$ from the experiment. One way is to set $0.29(2^{-1}Ri_0^{-1}-2)$ equal to the mean slope in figure 6 of DM at large $t$ (this is what we graphically did by choosing $Ri_0$ so that our graph in figure 1 agreed with the experimental curve in figure 6). The other way is to set $0.29(4Ri\sigma N^{-1})^{-1}$ equal to the experimental amplitude in figure 6. Denoting that amplitude by $\Delta Z_{\text{exp}}$, we have

$$\frac{0.29}{4Ri_0(\sigma/N)} = \Delta Z_{\text{exp}} \quad (\text{large } t),$$

$$Ri_0 \approx 0.3,$$

with $\Delta Z_{\text{exp}} \approx 0.36$. This value of $Ri_0$ is not far from the value 0.25 previously obtained by comparing the mean slope of theoretical and experimental energy decay. However, despite the favourable comparisons, there is a reason to question the validity of the comparison between our theory and the DM experiment – as we explain next.

*Validity of the comparison*

The preceding comparison between theory and experiment was based on the assumption that only a small part of the fluctuation energy is in the form of coherent gravity waves, and that most of the energy is due to random fluctuations (turbulence). The difficulty with this assumption, as pointed out by a referee, is that the reported mean motion (a coherent oscillation at frequency $\sigma \approx N$) is too weak to satisfy $Ri_0 \approx 0.25$. That value of $Ri_0$ is required in order for the theory to explain the experiment. Unless a source of small $Ri_0$ can be justified, the close agreement between theory and experiment may be just a remarkable coincidence. This discrepancy can be resolved if the observed fluctuation energy contains coherent gravity waves as well as turbulence. This case is given in the paragraph after next.

First let us consider a possibility, suggested by a referee, that the fluctuation energy is entirely in the form of coherent gravity waves. While possible, this assumption is also not free of drawbacks. For one thing, DM find that $k_z/k_x \approx 1.6$ (from $\sigma \approx 0.65N$), while at the same time $\langle w^2\rangle/\langle u^2\rangle \approx 1.1$. But coherent gravity waves with $k_z/k_x \approx 1.6$ will have $\langle w^2\rangle/\langle u^2\rangle \approx k_x^2/k_z^2 \approx 0.4$. Such a large discrepancy in $\langle w^2\rangle/\langle u^2\rangle$ cannot be attributed to experimental inaccuracies. Note, too, the spatial correlations shown in fig. 11 of DM. To us, the smooth and very broad correlations resemble random fluctuations more than they do coherent waves. Incidentally, a complication with deducing the influence of short-scale coherent waves on the measurements concerns whether such waves are 'standing' (e.g. $U = U_0 \cos(\boldsymbol{k}\cdot\boldsymbol{R})\cos(\sigma t+\phi)$) or 'travelling' ($U = U_0\cos(\boldsymbol{k}\cdot\boldsymbol{R}+\sigma t+\phi)$). In the latter case the experimental spatial average would give $\overline{U^2} = \frac{1}{2}U_0^2$, a constant, which does not oscillate with time, whereas the former case would give $\overline{U^2} = \frac{1}{2}U_0^2\cos^2(\sigma t+\phi)$, which does oscillate with time.

A third possibility is that part (say half) of the fluctuation energy is coherent gravity waves and half is turbulence, with the latter maintained by the coherent gravity waves according to (11). The value of $Ri_0$ for a coherent wave of that energy (about 0.12 $\text{cm}^2/\text{s}^2$ at $t = 1.6$ s) is $Ri_0 = N^2k^{-2}(0.12)^{-1} \approx 1$ if $k^2 \approx 1.2$. The turbulence-energy curve for $Ri_0 = 1$ is shown in figure 1. Let us add the energy $\overline{U^2}$ of the coherent wave to the turbulence energy curve for $Ri_0 = 1$ at $\phi = -\frac{1}{2}\pi$. This is done for three cases: ($a$) travelling waves $\overline{U^2} = \frac{1}{2}U_0^2$; ($b$) standing waves $\overline{U^2} = \frac{1}{2}U_0^2\cos^2(\sigma t+\phi)$; and ($c$) half-travelling–half-standing waves $\overline{U^2} = \frac{1}{4}U_0^2[1+\cos^2(\sigma t+\phi)$. The resulting curves are displayed in figure 4. Case ($c$) resembles the observation, and case ($b$) does to a lesser extent. Case ($a$) can be eliminated.

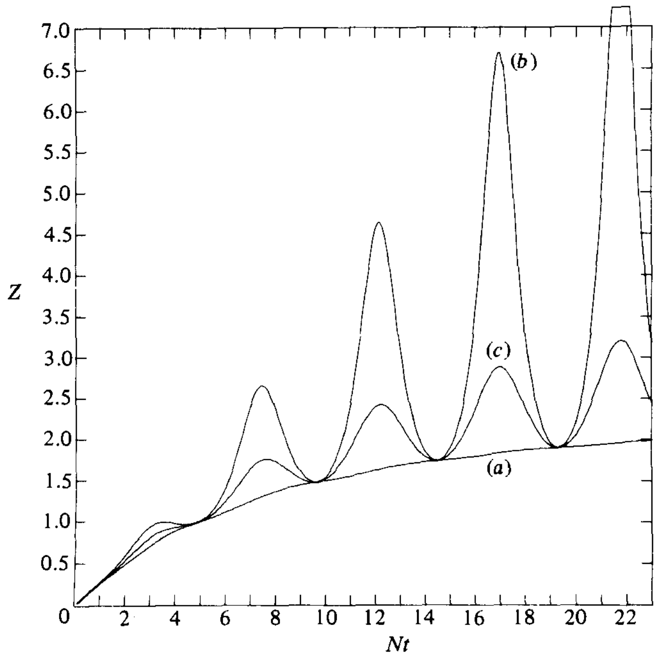


FIGURE 4. Theoretical kinetic-energy decay when half the energy is in turbulence as given by (11) with $Ri_0 = 1.0$ and half the energy is in coherent gravity waves. The values of $\sigma$ and $N$ are the same as in Dickey & Mellor (1980), and $Ri_0 = 1.0$. There are three cases shown: (*a*) travelling waves; (*b*) standing waves; and (*c*) half-travelling–half-standing waves.

A problem that remains is to reconcile the observed small anisotropy with the larger anisotropy of the wave kinetic energy. In this connection, a referee has suggested that the light refraction in the experiment might have contributed spuriously to the isotropy and randomness of the data.

## 4. Steady mean shear – maintenance of turbulence

At the suggestion of a referee, we consider the case of steady mean shear ($\sigma = 0$) and compare with previous work. In particular, a question that received much attention in the past is the maintenance of turbulence by a steady mean shear, i.e. what is the minimum value of background $Ri$, or of flux Richardson number $R_f$, required to prevent the turbulence from decaying to zero (e.g. Monin & Yaglom 1971, pp. 401–403; Yamada 1975). This value is referred to as a 'critical' value and denoted by $Ri_{cr}$. To determine $Ri_{cr}$, we solve (11) for a stationary condition at $\sigma = 0$. However, since the numerical constants 0.23 and 0.8 in (11) have some uncertainty, let us replace them by unspecified coefficients $a$ and $b$ respectively, and solve (the number 0.29 characterizes viscous dissipation and, we believe, has little uncertainty). Thus we rewrite (11) as

$$-\frac{\partial Z}{\partial (Nt)} = \frac{a}{Z^{-2}+b}\left(\frac{1}{Ri_0}-1\right)-0.29, \tag{13}$$

and solve for $\partial Z/\partial(Nt) = 0$ as $Nt \to \infty$, to obtain

$$Z^{-2} = (0.29)^{-1}\, a(Ri_0^{-1}-1)-b. \tag{14a}$$

The value of $Ri_{cr}$ is obtained by setting this $Z^{-2}$ equal to zero, the smallest value of $Z$ possible, and solving for $Ri_0$. The result is

$$Ri_{\text{cr}} = (1+0.29b/a)^{-1}. \tag{14b}$$

This relation yields $Ri_{\text{cr}} = 0.5$ for $b/a = 0.8/0.23$ as chosen in (11); $Ri_{\text{cr}} = 0.33$ for $b/a$ equal to twice that in (11); and $Ri_{\text{cr}} = 0.2$ for $b/a$ four times that in (11). However, from a practical consideration, these values of $Ri_{\text{cr}}$ are a kind of overestimate, because the turbulence energy $\frac{1}{2}q^2$ approaches zero as $Ri_0 \rightarrow Ri_{\text{cr}}$.

A comparison of (14*b*) with a popular eddy-viscosity model is easily made. That model has $b = 0$ (see §1) and yields $Ri_{\text{cr}} = 1$ regardless of the value of $a$ (the magnitude of the eddy coefficient). For $\sigma \neq 0$, the eddy-viscosity model overestimates the energy when $Ri_0$ is small ($< 0.5$), and, more seriously, leads to a divergence when $Ri_0 > 1$. It diverges because the buoyancy flux varies as $-aZ^2$ and removes energy excessively rapidly as $Z$ becomes large.

We expect that (11) and (14*a*) are comparable to what would be obtained by second-order models – such as the level 4 model of Mellor & Yamada (1974). The difference is that fewer equations are used, by virtue of buoyancy subrange theory, and separate equations for $\langle w^2 \rangle$ and $\langle u^2 \rangle$ are dispensed with by virtue of our weak anisotropy condition. These simplicities allow us to more readily concentrate on the physical significance of the few terms that do occur in the formalism and obtain a single closed equation for $q$. There are also fewer numerical constants.

## 5. Summary and conclusions

A calculation was made of the decay of turbulence in the presence of coherent gravity waves. With this model, turbulence decay is shown to satisfy a dimensionless equation that is a universal function of $Ri_0$ and wave frequency $\sigma$ – provided that the energy is non-dimensionally expressed in terms of the buoyancy wavenumber $k_{\text{B}}$, and the time is expressed in terms of the Brunt–Väisälä frequency $N$. A simplifying limitation of small anisotropy is also made. Solution of this equation shows that the influence of waves on turbulence can be divided into three different classes: (*a*) for $Ri_0 < 0.4$ the turbulence decay undergoes a sudden transition near $t(2\pi)^{-1} = 0.7N$ from a rapid monotonic to slow oscillating decay; (*b*) for $0.4 < Ri_0 < 0.8$ the decay resembles that of a neutral fluid; and (*c*) for $Ri_0 > 0.8$ the decay is more rapid than that of a neutral fluid. In all cases the coherent waves effectively decrease the rate of energy dissipation – the amount of decrease depending on the wave strain rate. Graphs of the theoretical decay including all three ranges of $Ri_0$ are given in figures 1–3.

The theoretical decay for $Ri_0 \approx 0.25$ in figure 1 was found to be in agreement with the observations (figure 6 of DM). However, there is a difficulty with this agreement because the reported coherent wave amplitudes are not large enough to yield the small $Ri_0$ value required by the theory. This difficulty is overcome if we account for 'course' spatial averaging. That is, since the measurements were averaged over a large volume (dimensions of $(12.7\text{ cm})^3$), the observed $q^2$ can be partly coherent waves and partly random fluctuations. The combination of (half) coherent wave energy and (half) random fluctuation energy leads to figure 4(*c*), and agrees with the observation including the apparent $Ri_0$ of the experiment. The possibility that the observed $q^2$ is entirely due to coherent waves can be discounted by the randomicity implied by the measured correlation function and by the near-isotropy of the measured kinetic-energy density.

An important question raised by Dickey & Mellor is why a sudden transition did not occur in horizontally towed grid decay experiments (e.g. Lin & Pao 1979; Lin & Veenhuizen 1974), even though internal waves did occur as observed by shadowgraphs. Does this imply that horizontal is different from vertical grid turbulence? A possible explanation is that, in the latter, fluid parcels of higher density are moved vertically by the grid into regions of lower density. This situation of higher-density fluid resting on top of lower-density fluid causes Helmholtz waves which oscillate near the Brunt–Väisälä frequency. Such waves are produced independently of the turbulence, and can have short scales on the order of the mesh size. The horizontal grids have less of a tendency to set up Helmholtz waves (although some are undoubtedly present). Hence, the magnitude of $Ri_0$ may be larger for horizontal than for vertical towed grids. A difference in $Ri_0$ could account for the different decays of the 2 kinds of experiments.

It is a pleasure to thank Carl Love for solving (11) by computer, and for programming the solution onto figure 1–4.

## Appendix: momentum flux in stratified fluids

The purpose of this Appendix is to evaluate $\langle u_z' u_H' \rangle$ in terms of $\nabla U$ and the energy components $\langle u_i'^2 \rangle$. A basic assumption we use is slow variation of average quantities on a Lagrangian timescale. Lumley & Newman (1977) have shown that such an assumption works very well for turbulence computations. Our calculation of $\langle u_z' u_H' \rangle$ begins with a formal Green-function solution of the Navier–Stokes equation (Weinstock 1981 – see equation (13)):

$$\begin{aligned} \boldsymbol{u}'(t) = {} & \int \mathrm{d}\boldsymbol{r}_1\, G(\boldsymbol{r},t;\boldsymbol{r}_1,0)\, \boldsymbol{u}'(\boldsymbol{r}_1,0) + \int_0^t \mathrm{d}t_1 \\ & \times \int \mathrm{d}\boldsymbol{r}_1\, G(\boldsymbol{r},t;\boldsymbol{r}_1,t_1) \left\{ \frac{-\nabla P'(\boldsymbol{r}_1,t_1)}{\rho_0} - \boldsymbol{u}'(\boldsymbol{r}_1,t_1)\cdot\nabla \boldsymbol{U} + \frac{\rho'(\boldsymbol{r}_1,t_1)\,\nabla P^0}{\rho_0^2} + \nu\nabla^2 \boldsymbol{u}(\boldsymbol{r}_1,t_1) \right\}, \end{aligned} \tag{A 1}$$

where the Green function $G(\boldsymbol{r},t;\boldsymbol{r}_1,t_1)$ is defined as the solution of

$$\left.\begin{aligned} \left(\frac{\partial}{\partial t} + \boldsymbol{u}'\cdot\nabla + \boldsymbol{U}\cdot\nabla\right) G(\boldsymbol{r},t;\boldsymbol{r}_1,t_1) = 0, \\ G(\boldsymbol{r}_1,t_1;\boldsymbol{r}_1,t_1) = \delta(\boldsymbol{r}-\boldsymbol{r}_1). \end{aligned}\right\} \tag{A 2}$$

Equation (A 1) can be verified by differentiating both its sides with respect to $t$ and then substituting (A 2) and (A 1) back into that differentiated equation. The quantity $G$ can also be viewed as an integrating factor of the Navier–Stokes equation. Fortunately, we will not need to know the explicit expression for $G$ in what follows. An expression for $\langle u_z' u_H' \rangle$ is obtained by multiplying the horizontal component of (A 1) by $u_z'$ and averaging. In so doing we neglect $\nu \nabla^2 \boldsymbol{u}'$ for large Reynolds number, and we also neglect the initial-value term $\boldsymbol{u}'(\boldsymbol{r}_1, 0)$, which is almost always done in turbulence theories and which can be justified except for very small values of $t$. The result is

$$\langle u_z'(t)\, u_H'(t) \rangle \approx \int \mathrm{d}t_1 \int \mathrm{d}\boldsymbol{r}_1 \left\langle u_z'(t)\, G(\boldsymbol{r},t;\boldsymbol{r}_1 t_1) \left[ -u_H'(t_1)\frac{\partial U_H}{\partial H} - u_z'(t_1)\frac{\partial U_H}{\partial z} - \frac{\partial P'(t_1)}{\rho_0 \partial H} \right] \right\rangle. \tag{A 3}$$

But $\int \mathrm{d}\boldsymbol{r}_1 \langle u_i'(t)\, G(\boldsymbol{r}_1, t; \boldsymbol{r}_1, t_1)\, u_j'(t_1)\rangle$ is precisely a Lagrangian correlation function and can be expressed as (e.g. Tennekes & Lumley 1972)

$$\int_0^t \mathrm{d}t_1 \int \mathrm{d}\boldsymbol{r}_1 \langle u_i'(t)\, G(\boldsymbol{r}, t; \boldsymbol{r}_1, t_1)\, u_j'(t_1)\rangle = \tau_\mathrm{L}\langle u_i'(t)\, u_j'(t)\rangle, \tag{A 4}$$

provided that $t \gg \tau_\mathrm{L}$ (which correspond to slow variation on a Lagrangian integral time). By definition, $\tau_\mathrm{L}$ is the Lagrangian integral timescale. We neglect the variation of $\tau_\mathrm{L}$ with direction (i.e. we assume $\tau_\mathrm{L}$ is the same for all $i$ and $j$ in (A 4)), which is satisfactory for the weakly anisotropic turbulence of the experiments of DM. The value of $\tau_\mathrm{L}$ for stable stratification has been previously determined (Weinstock 1978*b*), and is given at the end of this Appendix. Similarly to (A 4), it can be shown that

$$\int_0^t \mathrm{d}t_1 \int \mathrm{d}\boldsymbol{r}_1 \left\langle u_z'(t)\, G(\boldsymbol{r}, t; \boldsymbol{r}_1, t_1) \frac{\partial P'(t_1)}{\partial H}\right\rangle \approx \tau_\mathrm{L}\left\langle u_z'(t) \frac{\partial P'(t)}{\partial H}\right\rangle. \tag{A 5}$$

On substitution of (A 4) and (A 5) into (A 3) we obtain

$$\langle u_z' u_H'\rangle = -\tau_\mathrm{L}\left[\langle u_z' u_H'\rangle \frac{\partial U_H}{\partial H} + \langle u_z'^2\rangle \frac{\partial U_H}{\partial z} + \left\langle \frac{u_z'}{\rho_0}\frac{\partial P'}{\partial H}\right\rangle\right]. \tag{A 6}$$

Another expression for $\langle u_H' u_z'\rangle = \langle u_z' u_H'\rangle$ is obtained by multiplying the $z$ component of (A 1) by $u_H'$ and averaging. The result after making the same approximations as used for (A 4) is

$$\langle u_H' u_z'\rangle = -\tau_\mathrm{L}\left[\langle u_H' u_z'\rangle \frac{\partial W}{\partial z} + \langle u_H'^2\rangle \frac{\partial W}{\partial H} + \rho_0^{-1}\left\langle u_H' \frac{\partial P'}{\partial z}\right\rangle - \langle u_H' \rho'\rangle \frac{1}{\rho_0^2}\frac{\partial P^0}{\partial z}\right], \tag{A 7}$$

where we have used the fact $P^0$ varies with $z$. The pressure–velocity correlations in (A 6) and (A 7) have been modelled by Rotta (1951):

$$\rho_0^{-1}\left\langle u_z' \frac{\partial P'}{\partial H} + u_H' \frac{\partial P'}{\partial z}\right\rangle = -\frac{C_1}{\tau_\mathrm{L}}\langle u_z' u_H'\rangle + 0.4 e_0\left(\frac{\partial W}{\partial H} + \frac{\partial U_H}{\partial z}\right), \tag{A 8}$$

where $C_1 \approx 0.4$. Lumley & Newman (1977) suggest a slightly smaller value of $C_1$ (Weinstock 1981). The quantity $\rho_0^{-2}\langle u_H' \rho'\rangle$ $(\partial P^0/\partial z)$ is determined with use of

$$\rho'/\rho_0 = -\theta'/\theta_0 = -\int_0^t \mathrm{d}t_1\, u_z'(t_1)\,(\partial\theta_0/\partial z)\,\theta_0^{-1} \quad \text{and} \quad \partial P_0/\partial z = -gP_0,$$

where $\theta'$ and $\theta_0$ are fluctuating and mean potential temperatures respectively. We thus obtain

$$\rho_0^{-2}\langle u_H' \rho'\rangle \frac{\partial P_0}{\partial z} = -\frac{g}{\theta_0}\frac{\partial \theta_0}{\partial z}\int \mathrm{d}t'\, \langle u_H'(t)\, u_z'(t_1)\rangle = -\tau_\mathrm{L}\langle u_H' u_z'\rangle N^2. \tag{A 9}$$

This term is found to be smaller than the others in (A 6) and (A 7) and will be ignored. An expression for $\langle u_z' u_H'\rangle$ can be obtained by adding (A 6) and (A 7), substituting (A 8), and using $\langle u_z' u_H'\rangle = \langle u_H' u_z'\rangle$:

$$\begin{aligned} 2\langle u_z' u_H'\rangle = -\tau_\mathrm{L}\bigg[&\langle u_H'^2\rangle \frac{\partial W}{\partial H} + \langle u_z'^2\rangle \frac{\partial U_H}{\partial z} + \langle u_z' u_H'\rangle\left(\frac{\partial W}{\partial z} + \frac{\partial U_H}{\partial H}\right) \\ &- \frac{0.4}{\tau_\mathrm{L}}\langle u_z' u_H'\rangle + 0.4 e_0\left(\frac{\partial W}{\partial H} + \frac{\partial U_H}{\partial z}\right)\bigg]. \end{aligned} \tag{A 10}$$

But $\partial W/\partial z + \partial U_H/\partial H = 0$ for an incompressible flow, so that $\langle u_z' u_H' \rangle$ in (A 10) is given with $e_0 = 1.5 v_0^2$:

$$\langle u_z' u_H' \rangle = -\frac{\tau_L}{1.6}\left[\langle u_H'^2 \rangle \frac{\partial W}{\partial H} + \langle u_z'^2 \rangle \frac{\partial U_H}{\partial z}\right] - \frac{0.6 \tau_L v_0^2}{1.6}\left(\frac{\partial W}{\partial H} + \frac{\partial U_H}{\partial z}\right)$$

$$\approx \rho_L v_0^2 \left(\frac{\partial W}{\partial H} + \frac{\partial U_H}{\partial z}\right) \quad (\langle u_H'^2 \rangle \sim \langle u_z'^2 \rangle \sim v_0), \tag{A 11}$$

which is (3), as we set out to prove.

The buoyancy flux $\langle \rho' \boldsymbol{u}' \rangle \cdot \nabla P_0/\rho_0^2 = \langle \rho' u_z' \rangle (\partial P_0/\partial z) \rho_0^{-2}$ in (2) is evaluated with use of

$$\rho'/\rho_0 = -\theta'/\theta_0 = -\int_0^t \mathrm{d}t\, u_z'(t_1)\,(\partial \theta_0/\partial z)\,\theta_0^{-1}:$$

$$\langle \rho' \boldsymbol{u}' \rangle \cdot \nabla P^0/\rho_0^2 = \frac{g\,\partial \theta_0}{\theta_0\,\partial z}\int_0^t \mathrm{d}t_1 \langle u_z'(t)\, u_z(t_1) \rangle$$

$$= \tau_L \langle (u_z')^2 \rangle N^2 \approx \tfrac{1}{3}\tau_L q^2 N^2. \tag{A 12}$$

Finally we must evaluate $\tau_L$ approximately for stably stratified fluids. Such an evaluation was previously made by calculating the influence of buoyancy and random phased gravity waves on the dispersion of particles in a stratified fluid (Weinstock 1978*a*, Appendix B). The details will be found in that reference. Here we only present the derived approximate expression for $\tau_L$:

$$\tau_L \sim 0.35 \frac{k_0 v_0}{k_0^2 v_0^2 + 0.8 N^2} = \frac{0.35}{3^{\frac{1}{2}}} \frac{k_0 q}{k_0^2 v_0^2 + 0.8 N^2}, \tag{A 13}$$

where $k_0$ is a wavenumber characteristic of the energy-containing region of the turbulence spectrum. This wavenumber is defined with the following energy-spectral model $E(k)$ previously used by Reynolds (1976) and Comte-Bellot & Corrsin (1966):

$$E(k) = \alpha \epsilon_\nu^{\frac{2}{3}} k^{-\frac{5}{3}} \quad (k \geqslant k_0),$$

$$E(k) = \alpha \epsilon_\nu^{\frac{2}{3}} k_0^{-m-\frac{5}{3}} k^m \quad (k \leqslant k_0),$$

where $\epsilon_\nu$ is the dissipation rate due to molecular viscosity, $\alpha \approx 1.5$ is the Kolmogoroff constant, and $m \geqslant 0$ is an adjustable numerical parameter (Reynolds argued that $m \approx 2$ is supported by semi-empirical considerations). For this model, $q^2$, $\epsilon_\nu$ and $k_0$ are related by

$$q^2 = 2\int \mathrm{d}k\, E(k) = 2\alpha \epsilon_\nu^{\frac{2}{3}} k_0^{-\frac{2}{3}} [1 + \tfrac{3}{2}(m+1)^{-1}],$$

and the variation of $q^2$ with $m$ is very small. We will use a median value of $m = 3$. Hence

$$\epsilon_\nu = 0.083 k_0 q^3 \quad (m = 3), \tag{A 14}$$

which will be needed to discuss the data of DM. Note that $\tau_L$ reduces properly to the correct neutral limit as $N \to 0$ (e.g. Tennekes & Lumley 1972). It also reduces to the correct strongly stratified limit as $N \to \infty$, in agreement with Lilly *et al.* (1974), Caldwell *et al.* (1980), Weinstock (1978*b*) and Zimmerman & Loving (1975). The two numerical factors in (A 12) are not exact, but are correct to within a factor of about 2. The important point we wish to make is that changes in these constants will not cause qualitative changes in our equations for $q^2$, nor in the theoretical curves of figure 1–3. Rather, an increase of the constant 0.35 will cause a less than proportionate decrease of the curves in figure 1–3 at large $t$, and an increase of the constant 0.8 will cause a less than proportionate increase of those curves at large $t$. The ratio of these two constants does influence the value of $Ri_{cr}$ in (13$b$).

## REFERENCES


BATCHELOR, G. K. & TOWNSEND, A. A. 1948 Decay of isotropic turbulence in the initial period. *Proc. R. Soc. Lond.* A **193**, 539–566.

BOUCHER, R. J. 1974 Evaluation of clear air turbulence detection by ground based radars, special rawinsondes and aircraft, 1967–1971. AFCRL-TR-74-0489, *Air Force Cambridge Research Lab., Hanscomb, MA*.

BRITTER, R. E., HUNT, J. C. R., MARSH, C. L. & SNYDER, W. H. 1983 The effects of stable stratification on turbulent diffusion and the decay of grid turbulence. *J. Fluid Mech.* **127**, 27–44.

CALDWELL, D. R., DILLON, T. M., BRUBAKER, J. M., NEWBERGER, P. A. & PAULSON, C. A. 1980 The scaling of vertical temperature gradient spectra. *J. Geophys. Res.* **85**, 1917–1924.

COMTE-BELLOT, G. & CORRSIN, S. 1966 The use of a contraction to improve the isotropy of gridgenerated turbulence. *J. Fluid Mech.* **25**, 667–687.

DICKEY, T. D. & MELLOR, G. L. 1980 Decaying turbulence in neutral and stratified fluids. *J. Fluid Mech.* **99**, 37–48.

DUTTON, J. A. & PANOFSKY, H. A. 1970 Clear air turbulence: a mystery may be unfolding. *Science* **167**, 937–944.

EINAUDI, F., LALAS, D. P. & PERON, G. E. 1978–1979 The role of gravity waves in the tropospheric processes. *Pure Appl. Geophys.* **117**, 627–663.

FUA, D., CHIMONAS, G., EINAUDI, F. & ZEMAN, O. 1982 An analysis of wave–turbulence interaction. *J. Atmos. Sci.* **39**, 2450–2463.

GARGETT, A. E., HENDRICKS, P. J., SANFORD, T. B., OSBORN, T. R. & WILLIAMS, A. J. 1981 A composite spectrum of vertical shear in the upper ocean. *J. Phys. Oceanogr.* **11**, 1258–1271.

KLEMP, J. B. & LILLY, D. K. 1975 The dynamics of wave-induced downslope winds. *J. Atmos. Sci.* **32**, 320–339.

LANGE, R. E. 1974 Decay of turbulence in stratified salt water. PhD thesis, University of California, San Diego.

LILLY, D. K., WACO, D. E. & ADLEFANG, S. I. 1974 Stratospheric mixing estimated from high altitude turbulence measurements. *J. Appl. Meteor.* **13**, 488–493.

LIN, J. T. & PAO, Y. H. 1979 Waves in stratified fluids. *Ann. Rev. Fluid Mech.* **11**, 317–388.

LIN, J. T. & VEENHUIZEN, S. D. 1974 Measurements of the decay of grid generated turbulence in a stably stratified fluid (abstract). *Bull. Am. Phys. Soc.* **19**, 1142–1143.

LINDEN, P. F. 1980 Mixing across a density interface produced by grid turbulence. *J. Fluid Mech.* **100**, 691–703.

LUMLEY, J. L. 1964 The spectrum of nearly inertial range turbulence in a stably stratified fluid. *J. Atmos. Sci.* **21**, 99–102.

LUMLEY, J. L. & NEWMAN, G. R. 1977 The return to isotropy of homogeneous turbulence. *J. Fluid Mech.* **82**, 161–178.

LUMLEY, J. L. & PANOFSKY, H. A. 1964 *The Structure of Atmospheric Turbulence*. Interscience.

MELLOR, G. L. & YAMADA, T. 1974 A hierarchy of turbulence closure models for planetary boundary layers. *J. Atmos. Sci.* **31**, 1791–1806.

MONIN, A. S. & YAGLOM, A. M. 1971 *Statistical Fluid Mechanics*, pp. 397–398. MIT Press.

PAO, Y. H. 1973 Measurements of internal waves and turbulence in two-dimensional stratified shear flows. *Boundary-Layer Meteor.* **5**, 177–193.

REYNOLDS, W. C. 1976 Computation of turbulent flows. *Ann. Rev. Phys.* **8**, 183–208.

ROTTA, J. 1951 Statistische Theorie nichthomogener Turbulenz. 1. Mitteilung. *Z. Phys.* **129**, 547–572.

RÜSTER, R., ROTTGER, J. & WOODMAN, R. F. 1978 Radar measurements of waves in the lower troposphere. *Geophys. Res. Lett.* **5**, 555–558.

Tennekes, H. & Lumley, J. L. 1972 *A First Course in Turbulence.* MIT Press.

Van Zandt, T. E., Green, J. L. & Clark, W. L. 1979 Buoyancy waves in the troposphere: Doppler radar observations and a theoretical model. *Geophys. Res. Lett.* **6**, 429–432.

Weinstock, J. 1978*a* On the theory of turbulence in the buoyancy subrange of stably stratified flows. *J. Atmos. Sci.* **35**, 634–649.

Weinstock, J. 1978*b* Vertical turbulent diffusion in a stably stratified fluid. *J. Atmos. Sci.* **35**, 1022–1027.

Weinstock, J. 1981 Theory of the pressure–strain-rate correlation for Reynolds-stress turbulence closures. Part 1. Off-diagonal element. *J. Fluid Mech.* **105**, 369–395.

Yamada, T. 1975 The critical Richardson number and the ratio of the eddy transport coefficients obtained from a turbulence closure model. *J. Atmos. Sci.* **32**, 926–933.

Zimmerman, S. P. & Loving, N. V. 1975 Turbulence dissipation and diffusivities in the stratosphere using Richardson's technique. In. *The Natural Stratosphere of 1974; CIAP Monograph* 2, *Dept. Transportation* TST-75-51, pp. 6-93–6-103.